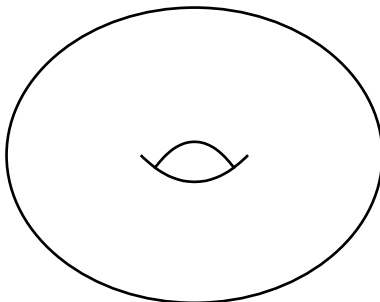
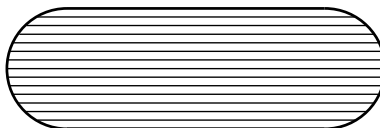


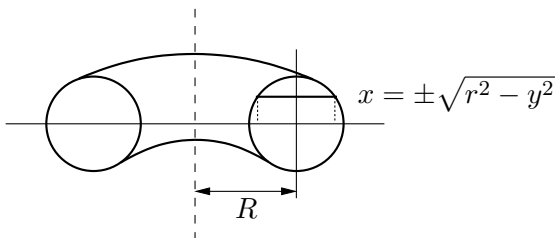
Computing the volume of a torus.



Washer Method For this method we need to slice the torus like one would slice a bagel.



We consider the following image to determine the large and small radius of each “washer” that we created by the above slicing.



Large radius: $R + \sqrt{r^2 - y^2}$
Small radius: $R - \sqrt{r^2 - y^2}$

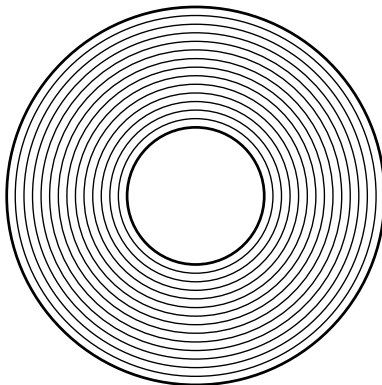
Volume:

$$\begin{aligned} V_1 &= \pi \int_{-r}^r (R + \sqrt{r^2 - y^2})^2 - (R - \sqrt{r^2 - y^2})^2 dy \\ &= \pi \int_{-r}^r (R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2) - (R^2 - 2R\sqrt{r^2 - y^2} + r^2 - y^2) dy \\ &= \pi \int_{-r}^r 4R\sqrt{r^2 - y^2} dy \\ &= 4\pi R \int_{-r}^r \sqrt{r^2 - y^2} dy \end{aligned}$$

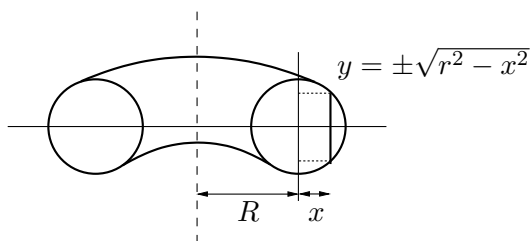
Using trigonometric substitution (or the tables in the back of the book) we find,

$$\begin{aligned}
 &= 4\pi R \left(\frac{y}{2} \sqrt{r^2 - y^2} + \frac{r^2}{2} \sin^{-1} \frac{y}{r} \right) \Big|_{-r}^r \\
 &= 4\pi R \left(\left(\frac{r}{2} \sqrt{r^2 - r^2} + \frac{r^2}{2} \sin^{-1} \frac{r}{r} \right) - \left(\frac{-r}{2} \sqrt{r^2 - (-r)^2} + \frac{r^2}{2} \sin^{-1} \frac{-r}{r} \right) \right) \\
 &= 4\pi R \left(\left(\frac{r^2}{2} \sin^{-1} 1 \right) - \left(\frac{r^2}{2} \sin^{-1}(-1) \right) \right) \\
 &= 4\pi R \left(\left(\frac{r^2}{2} \frac{\pi}{2} \right) - \left(\frac{r^2}{2} \frac{-\pi}{2} \right) \right) \\
 &= (2\pi R)(\pi r^2)
 \end{aligned}$$

Shells Method This time, we slice in concentric rings.



We are interested in the radius and the height of each cylindrical shell.



Radius: $x - (-R) = x + R$

Height: $2\sqrt{r^2 - x^2}$

Volume:

$$\begin{aligned}
 V_2 &= 2\pi \int_{-r}^r (x + R)(2\sqrt{r^2 - x^2}) dx \\
 &= 4\pi \left(\int_{-r}^r x\sqrt{r^2 - x^2} dx + \int_{-r}^r R\sqrt{r^2 - x^2} dx \right)
 \end{aligned}$$

For the first integral, we try the substitution $u = r^2 - x^2$, thus $du = -2x dx$. Modifying the limits we find $x = -r \mapsto u = 0$ and $x = r \mapsto u = 0$ thus,

$$\int_{-r}^r x \sqrt{r^2 - x^2} dx = \int_0^0 \frac{\sqrt{u}}{-2} du = 0.$$

The integral is zero because the upper and lower limits are the same. Note that the same substitution does not work in the second integral $\int_{-r}^r R \sqrt{r^2 - x^2} dx$ since $du = -2x dx$ and there is no x to match the $x dx$.

However, we now have,

$$V_2 = 4\pi \left(\int_{-r}^r R \sqrt{r^2 - x^2} dx \right)$$

which matches the fourth step in the computation of the volume when we used the washer method. Thus, $V = (2\pi R)(\pi r^2)$.

Translated Axis In class we constructed a formula,

$$V_3 = 2\pi \int_{R-r}^{R+r} X \sqrt{r^2 - (X - R)^2} dX$$

by considering the y -axis to pass through the center of the torus (the axis of rotation). If, to V_3 , we apply the substitution $u = X - R$, then $du = dX$, $X = u + R$, and

$$V_3 = 2\pi \int_{-r}^r (u + R) \sqrt{r^2 - u^2} du$$

which is exactly the integral we started with in V_2 .

Other Problems

In these computations the *method of integration was dictated to us*. Since we had a particular method in mind we were forced to choose horizontal or vertical slices. Typically, however, we are presented with functions and the method used is at our discretion. In these cases it makes sense to *first* choose an orientation for our slices — vertical slices if the equations are easily solved for y , horizontal slices if the equations are easily solved for x . Only then should we decide which method to use based on whether our slices are parallel (shells) or perpendicular (washers) to the axis of rotation.